Reliability allocation in series-parallel system

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Abstract

In order to improve system reliability, designers may introduce in a system different technologies in parallel. When each technology is composed of components in series, the architecture belongs to the series-parallel systems. This type of system has not been studied as much as the parallel-series ones. There exists no method dedicated to the reliability allocation in series-parallel systems with different technologies. We propose in this paper a study of this reliability allocation problem. We first develop the optimal resolution of a one stage-problem, assuming that the cost functions satisfy a predefined condition. This condition is satisfied by most of the cost functions of the literature. Then, we establish a convexity condition on the component cost functions in order to obtain a global optimal solution. As it is impossible to verify this condition for the cost functions of the literature; we use the one-stage result to provide an approximation method. This approach use the least square method in order to obtain an approximation of the cost functions of each technology and the Lagrangian relaxation approach which gives an approximate solution of the global problem. Finally, we propose a detailed example and numerical experiments.

1 Introduction

One of the undeniable steps in the design of multicomponent systems is the problem of using the available resources in the most effective way so as to maximize the overall system reliability, or so as to minimize the consumption of resources while achieving specific reliability goals (Tzafestas 1980). The diversity of system structures, resource constraints, and options for reliability improvement has led to the construction and analysis of several optimization models (Kuo and Prasad 2000). Surveys on reliability optimization have regularly appeared (e.g. Tzafestas 1980 and Tillman *et al.* 1980). One of the most recent publications on this subject is the book of Kuo *et al.* (2001). According to these authors, several investigations have been done for the optimization of parallel-series systems, but very few are devoted to the series-parallel systems. Most of theses works deal with redundancy allocation as the method developed by Jensen (1970) for series-parallel-series networks using dynamic programming. He assumed that the studied systems are composed of blocks in series and each block is composed of identical technologies in parallel.
In this paper, we are interested in the reliability allocation problem in which the reliability of the components have to be determined in order to minimize the consumption of a resource under a reliability constraint, in a series-parallel systems. This problem arise when different technologies which may have the same function are used in parallel. We propose, in the case of convex cost functions, a condition for obtaining the optimality of the solution. We show that we are able to apply this condition to the case of Tillman functions, but this gives a too restrictive condition. An approximation resolution method, which is based on the least square method and the Lagrangian relaxation one, is then developed and applied to the Truelove function. This paper is organized as follows. In section 2, we present the problem description, with the assumptions and mathematical formulation. Section 3 is dedicated to the one-stage problem. In section 4, we present the global resolution. Section 5 describes the numerical experiments for investigating the performances of the method.

2 Problem description

A series-parallel system is composed of k subsystems in parallel, where each subsystem $i = 1, ...k$ is composed of n_i components in series. Let r_{ij} the reliability of the j^{th} component of subsystem i ($i = 1, ...k$ and $j = 1, ...n_i$). The reliability of the system is $R_s = 1 - \prod_{i=1}^{k}(1 - \prod_{j=1}^{n_i} r_{ij})$. We assume that c_{ij} is the cost associated to r_{ij} with $c_{ij} = f_{ij}(r_{ij})$. The total cost of the system is $C_s = \sum_{i=1}^{k}\sum_{j=1}^{n_i} f_{ij}(r_{ij})$. We assume that the decision variables r_{ij} may be any real value between 0 and 1. The problem of cost minimization with reliability constraint is :

$$\min\{\sum_{i=1}^{k}\sum_{j=1}^{n_i} f_{ij}(r_{ij}) \text{ so that } 1-\prod_{i=1}^{k}(1-\prod_{j=1}^{n_i} r_{ij}) = R_{min} \text{ and } 0 < r_{ij} < 1, \text{with } \ i = 1, ...k \text{ and } j = 1, ...n_i\} \quad (1)$$

3 Optimal resolution of a one-stage problem

Assume that we know the reliability target R_i for the subsystem $i = 1, ...k$. The subproblem corresponding to a subsystem i is to find the reliability allocation of the components in a series system.

$$\min\{\sum_{j=1}^{n_i} f_{ij}(r_{ij}) \text{ so that } \prod_{j=1}^{n_i} r_{ij} = R_i \text{ and } 0 < r_{ij} < 1 \quad \text{with } j = 1, ...n_i\} \quad (2)$$

Let $y_{ij} = \ln(r_{ij})$, ie. $r_{ij} = \exp(y_{ij})$, and $h_{ij}(y_{ij}) = f_{ij}(r_{ij})$. Let $y_{in_i} = \ln(R_i) - \sum_{j=1}^{n_i-1} y_{ij}$. The problem is:

$$\min\{\sum_{j=1}^{n_i-1} h_{ij}[y_{ij}] + h_{in_i}[\ln(R_i) - \sum_{j=1}^{n_i-1} y_{ij}] / \sum_{j=1}^{n_i-1} y_{ij} > \ln(R_i), \ y_{ij} \in]-\infty; 0[, \ j = 1, ..., n_i - 1\} \quad (3)$$

We assume that all the functions h_{ij} are strictly convex, which is the case for most cost functions in the literature. Then, the objective function to minimize is convex. At the optimum, the partial derivations are equal to zero. This gives the following equalities:

$$h'_{i1}(y_{i1}) = h'_{i2}(y_{i2}) = ... = h'_{in_i}[\ln(R_i) - \sum_{j=1}^{n_i-1} y_{ij}] \quad (4)$$

Thus, in order to solve a one stage problem, we have to solve the following system, where A is a positive value, which depends on the reliability target R_i:

$$\begin{cases} h'_{ij}(y_{ij}) = A(R_i) \quad j = 1, ..., n_i \\ \sum_{j=1}^{n_i} y_{ij} = \ln(R_i) \end{cases} \quad (5)$$

4 Resolution of the multi-stages problem

4.1 Optimal resolution: condition of convexity

Let $Y_i = \ln(1 - R_i)$. From the system (5), which gives the reliability allocation for an objective R_i for subsystem i, we define $H_i(Y_i) = \min\{\sum_{j=1}^{n_i} h_{ij}(y_{ij}(Y_i))$ so that $y_{ij}(Y_i) = h'^{-1}_{ij}(\varphi(Y_i))$, with $\varphi(Y_i) = A(R_i)$, and so that $\sum_{j=1}^{n_i} y_{ij}(Y_i) = \ln(1 - \exp(Y_i))\}$. The global problem is:

$$\min\{\sum_{i=1}^{k} H_i(Y_i) \text{ so that } \sum_{i=1}^{k} Y_i = \ln(1 - R_{min}) \text{ and } Y_i \in]-\infty; 0[, \ \forall \ i = 1, ...k\} \quad (6)$$

In order to solve this problem optimally, we should have functions H_i convex ($i = 1, ...k$). So, we have to find conditions on the functions h_{ij}, so that $H_i(Y_i)$ are convex. We define $g(z) = \sum_{j=1}^{n_i} h'^{-1}_{ij}(z)$. We have shown that the function $H_i(Y_i)$ is convex, ie. $H''_i(Y_i) > 0$ if:

$$\varphi(Y_i)g'(\varphi(Y_i)) < \exp(Y_i) \quad \Leftrightarrow \quad g^{-1}(\ln(R_i))g'(g^{-1}(\ln(R_i))) < 1 - R_i \quad \forall \ 1 > R_i > 0 \quad (7)$$

This gives a condition on the cost functions h_{ij} (i.e. f_{ij}) for obtaining the optimality of the reliability allocation. If this condition is satisfied, we are able to apply a result of Elegbede *et al.* (2003):

$$Y_i = \frac{\ln(1-R_{min})}{k} \ \forall\, i = 1,...,k \quad \Leftrightarrow \quad R_i = 1 - \sqrt[k]{1-R_{min}} \ \forall\, i = 1,...,k \tag{8}$$

which gives the optimal reliability allocation for each subsystem. In the literature, the most used cost function satisfy the assumption on the convexity of the h_{ij} functions. In order to verify the convexity condition (7), we need to express the functions $g^{-1}(\alpha)$ and $g'(\alpha)$. We are able to do it only for the Tillman functions ($f_{ij}(r_{ij}) = a_{ij}(r_{ij})^{b_{ij}}$, with $a_{ij} > 0, 1 > b_{ij} > 0$) (Elegbede 2000), and we obtain the following condition for having the functions H_i convex:

$$R_i < 1 - \sum_{j=1}^{n_i} \frac{1}{b_{ij}} \tag{9}$$

The parameters b_{ij} are between zero and one (Elegbede 2000). Then, $1/b_{ij} > 1$, and implies $R_i < 0$ which is impossible for a reliability. We finally cannot exploit the convexity condition for the Tillman functions.

4.2 Approximation method : the Truelove functions

Let $Z_i = \ln(R_i)$, for $i = 1,...k$. The cost function of subsystem i, for a reliability Z_i required is $h_i(Z_i) = \min\{\sum_{j=1}^{n_i} h_{ij}(y_{ij}(Z_i))$ so that $y_{ij}(Z_i) = h_{ij}^{-1\prime}(A(Z_i))$ and $\sum_{j=1}^{n_i} y_{ij}(Z_i) = Z_i\}$. The global problem is:

$$\min\{\sum_{i=1}^{k} h_i(Z_i) \text{ so that } 1 - \prod_{i=1}^{k}(1-\exp(Z_i)) = R_{min} \text{ and } Z_i \in]-\infty;0[, \quad i = 1,...k\} \tag{10}$$

We are enable to solve this problem optimally because, in the one hand, this problem is not convex, and on the other hand, we do not know the functions $h_i(Z_i)$. We know that the functions h_{ij} are strictly convex and increasing functions, so as $Z_i = \sum_{j=1}^{n_i} y_{ij}$, the function $h_i(Z_i)$ is strictly convex and increasing.
We use the least square method to approximate the functions $h_i(Z_i)$. With the system (5), we generate $h_i(Z_i) = \min_{y_{i1},...,y_{in_i}}\{\sum_{j=1}^{n_i} h_{ij}(y_{ij})\}$ for M values R_i^t, with $1 \le t \le M$ so that $R_i^t = R_i^{t-1} + \rho$, where $0 < \rho < 1$ and $0 < R_i^t < 1$. Then, we have M couple of values $(R_i^t, h_i(\ln(R_i^t)))$, for $i = 1,...k$. We know that the functions h_i are strictly convex and increasing from $]-\infty, 0[$ to $]0,\infty[$. We are here interested in the Truelove function : $f_{ij}(r_{ij}) = \frac{a_{ij}}{(1-r_{ij})^{b_{ij}}}$, with $a_{ij} > 0, b_{ij} > 0$ (Elegbede 2000). Using the least square method, we may approximate $h_i(Z_i)$ by a function $\hat{h}_i(Z_i) = \frac{\alpha_i}{Z_i^n} + \frac{\beta_i}{Z_i^{n-1}} + ... + \frac{\gamma_i}{Z_i} + \delta_i$, with $n > 0$ an integer. Thus, we have to solve the following problem:

$$\min\{\sum_{i=1}^{k} \hat{h}_i(Z_i) \text{ so that } 1 - \prod_{i=1}^{k}(1-\exp(Z_i)) = R_{min} \text{ and } Z_i \in]-\infty;0[, \ i = 1,...k\} \tag{11}$$

This problem is not convex. We apply the Lagrangian relaxation approach in order to obtain a local optimum. Let $Z = (Z_1,...,Z_k)$, $H(Z) = \sum_{i=1}^{k} \hat{h}_i(Z_i)$ and $G(Z) = 1 - \prod_{i=1}^{k}(1-\exp(Z_i)) - R_{min}$. We define the $L(Z,\lambda) = H(Z) - \lambda G(Z)$. We have a local minimum if $\nabla H(Z) + \lambda \nabla G(Z) = 0$. This gives the following system to solve to obtain a solution for the reliability allocation problem for the subsystems:

$$\begin{cases} \hat{h}_i'(\ln(R_i))(\frac{1}{R_i} - 1) = -B \quad i = 1,...,k \\ \prod_{i=1}^{k}(1-R_i) = 1 - R_{min} \end{cases} \tag{12}$$

5 Numerical experiments with the Truelove functions

We present 4 examples (with $k = 2,3,4,5$) in table 1. For each subsystem $i = 1,...k$, $n_i = 2$. The reliability objective which is 0.97, is obtained for all the examples, with $n = 3$ for the least square method. The results are presented in table 2, with the running time, indicated in CPU seconds (problems tested on a Pentium 4). The more the number of technologies increase, the more the probability that their exist less expensive configurations with less technologies increase.

Table 1: Data of the 4 Truelove examples

i,j	1,1	1,2	2,1	2,2	3,1	3,2	4,1	4,2	5,1	5,2
a_{ij}	1.7	2.5	1.58	2.1	2.7	2.4	1.9	2.90	2.03	3.7
b_{ij}	1.2	1.5	1.29	1.08	1.2	1.54	1.07	1.51	1.28	1.02

Table 2: Results for the 4 Truelove examples

k	C_1	C_2	C_3	C_4	C_5	C	time
2	72.79	87.70				160.49	0.156
3	38.79	50.23	36.20			125.21	0.234
4	25.83	35.70	23.06	27.63		111.62	0.312
5	18.85	27.95	15.33	20.38	23.31	108.82	0.375

6 Conclusion

Reliability allocation in series-parallel systems is one of the less studied problems in reliability optimization. We have proposed in this paper a study of this reliability allocation problem. First, we have developed the optimal resolution of a one stage-problem, assuming the cost functions satisfy a convexity condition. Then, we have established a convexity condition on the component cost functions in order to obtain the global optimal solution, but we cannot use it. We have provide an approximation method which use the least square method and the Lagrangian relaxation approach. Now, we analyze the performances of the method with other cost functions and more complex systems. This method may be also used to elaborate resolution approaches for more general structure of systems.

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